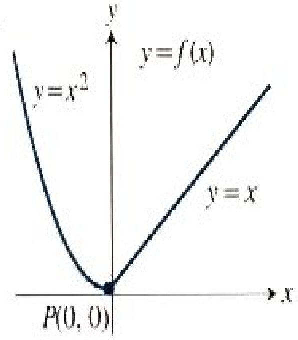
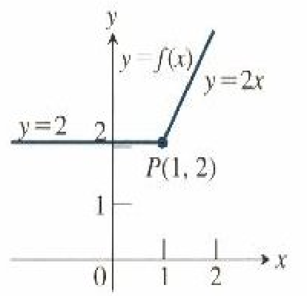
This worksheet serves as an additional exercise to complement the lesson and the examples given. Worksheets may take more than one day to complete. If you are stuck, read over the notes taken in class, see the additional examples from the website, work with fellow students, and come to the tutoring lessons after school. Complete solutions to all questions are available on the website. When in doubt do more!

In Exercises 1-4, compare the right-hand and left-hand derivatives to show that the function is not differentiable at the point *P*.

1.



2.



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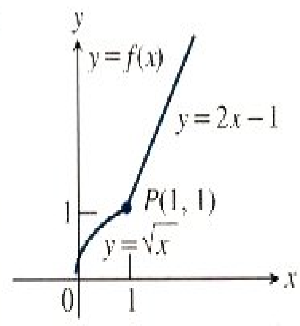

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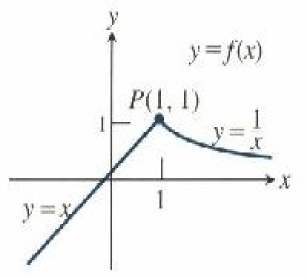
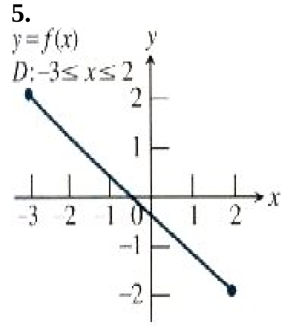


In Exercises 5-10, the graph of a function over a closed interval *D* is given. At what points does the function appear to be

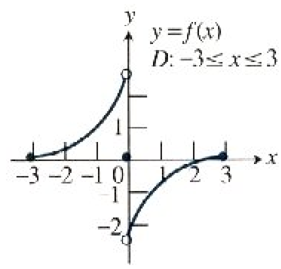
(a) differentiable?
(b) continuous but not differentiable?
(c) neither continuous nor differentiable?

5.

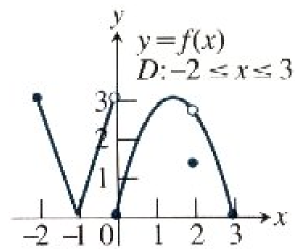


6.

7.

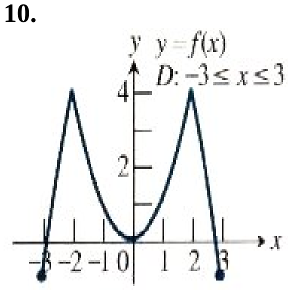


8.



9.

10.



In Exercises 11-16, the function fails to be differentiable at $x = 0$. Tell whether the problem is a corner, a cusp, a vertical tangent, or a discontinuity.

11. $y = \begin{cases} \tan^{-1} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$

12. $y = x^{4/5}$

13. $y = x + \sqrt{x^2} + 2$

14. $y = 3 - \sqrt[3]{x}$

15. $y = 3x - 2|x| - 1$

16. $y = \sqrt[3]{|x|}$

In Exercies17-22, find all values of *x* for which the function is differentiable.

17. $f(x) = \dfrac{x^3 - 8}{x^2 - 4x - 5}$

18. $h(x) = \sqrt[3]{3x - 6} + 5$

19. $P(x) = \sin(|x|) - 1$

20. $Q(x) = 3\cos(|x|)$

21. $g(x) = \begin{cases} (x+1)^2, & x \leq 0 \\ 2x + 1, & 0 < x < 3 \\ (4 - x)^2, & x \geq 3 \end{cases}$

22. $C(x) = x|x|$

1.
Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0^-}\frac{h^2-0}{h}$$
$$=\lim_{h\to 0^-} h$$
$$=0$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0^+}\frac{h-0}{h}$$
$$=\lim_{h\to 0^+} 1$$
$$=1$$

Since 0≠1, the function is not differentiable at the point *P*.

2.
Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0^-}\frac{2-2}{h}$$
$$=\lim_{h\to 0^-} 0$$
$$=0$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0^+}\frac{2(1+h)-2}{h}$$
$$=\lim_{h\to 0^+} 2$$
$$=2$$

Since 0≠2, the function is not differentiable at the point *P*.

3.
Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0^-}\frac{\sqrt{1+h}-1}{h}$$
$$=\lim_{h\to 0^-}\frac{\left(\sqrt{1+h}-1\right)\left(\sqrt{1+h}+1\right)}{h\left(\sqrt{1+h}+1\right)}$$
$$=\lim_{h\to 0^-}\frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)}$$
$$=\lim_{h\to 0^-}\frac{1}{\sqrt{1+h}+1}$$
$$=\frac{1}{2}$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0^+}\frac{\left[2(1+h)-1\right]-1}{h}$$
$$=\lim_{h\to 0^+}\frac{2h}{h}$$
$$=\lim_{h\to 0^+} 2$$
$$=2$$

Since $\frac{1}{2}\neq 2$, the function is not differentiable at the point *P*.

4.
Left-hand derivative:

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0^-}\frac{(1+h)-1}{h}$$
$$=\lim_{h\to 0^-} 1$$
$$=1$$

Right-hand derivative:

5(a). All points in [3, 2]

(b). None

(c). None

6(a). All points in [-2, 3]

(b). None

(c). None

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0^+}\frac{\frac{1}{1+h}-1}{h}$$
$$=\lim_{h\to 0^+}\frac{1-(1+h)}{h(1+h)}$$
$$=\lim_{h\to 0^+}\frac{-h}{h(1+h)}$$
$$=\lim_{h\to 0^+}-\frac{1}{1+h}$$
$$=-1$$

Since 1≠-1, the function is not differentiable at the point P.

7(a). All points in [-3, 3] except $x = 0$

(b). None

(c). $x = 0$

8(a). All points in [-2, 3] except $x = -1, 0, 2$

(b). $x = -1$

(c). $x = 0$, $x = 2$

9(a). All points in [-1, 2] except $x = 0$

(b). $x = 0$

(c). None

10(a). All points in [-3, 3] except $x = -2, 2$

(b). $x = -2$, $x = 2$

(c). None

11.
Since $\lim_{x\to 0}\tan^{-1}x=\tan^{-1}0=0\neq y(0)$, the problem is a discontinuity.

12.

$$\lim_{h\to 0^-}\frac{y(0+h)-y(0)}{h}$$
$$=\lim_{h\to 0^-}\frac{h^{\frac{4}{5}}}{h}$$
$$=\lim_{h\to 0^-}\frac{1}{h^{\frac{1}{5}}}=-\infty$$
$$\lim_{h\to 0^+}\frac{y(0+h)-y(0)}{h}$$
$$=\lim_{h\to 0^+}\frac{h^{\frac{4}{5}}}{h}$$
$$=\lim_{h\to 0^+}\frac{1}{h^{\frac{1}{5}}}=\infty$$

The problem is a cusp.

13. Note that

$y = x + \sqrt{x^2} + 2$

$= x + |x| + 2$

$= \begin{cases} 2, & x \le 0 \\ 2x + 2, & x > 0 \end{cases}$

$\lim_{h \to 0^-} \frac{y(0+h) - y(0)}{h}$

$= \lim_{h \to 0^-} \frac{2-2}{h}$

$= \lim_{h \to 0^-} 0$

$= 0$

$\lim_{h \to 0^+} \frac{y(0+h) - y(0)}{h}$

$= \lim_{h \to 0^+} \frac{(2h+2) - 2}{h}$

$= \lim_{h \to 0^+} 2$

$= 2$

The problem is a corner.

14.

$\lim_{h \to 0} \frac{y(0+h) - y(0)}{h}$

$= \lim_{h \to 0} \frac{\left(3 - \sqrt[3]{h}\right) - 3}{h}$

$= \lim_{h \to 0} -\frac{\sqrt[3]{h}}{h}$

$= \lim_{h \to 0} \left(-\frac{1}{h^{\frac{2}{3}}} \right)$

$= -\infty$

The problem is a vertical tangent.

15.

Note that

$y = 3x - 2|x| - 1$

$= \begin{cases} 5x - 1, & x \le 0 \\ x - 1, & x > 0 \end{cases}$

$\lim_{h \to 0^-} \frac{y(0+h) - y(0)}{h}$

$= \lim_{h \to 0^-} \frac{(5h-1) - (-1)}{h}$

$= \lim_{h \to 0^-} 5$

$= 5$

$\lim_{h \to 0^+} \frac{y(0+h) - y(0)}{h}$

$= \lim_{h \to 0^+} \frac{(h-1) - (-1)}{h}$

$= \lim_{h \to 0^+} 1$

$= 1$

The problem is a corner.

16.

17.

Find the zeros of the denominator.

$x^2 - 4x - 5 = 0$

$(x+1)(x-5) = 0$

$x = -1$ or $x = 5$

The function is a rational function, so it is differentiable for all x in its domain: all reals except $x = -1, 5$.

18.

The function is differentiable except possibly where 3x – 6 = 0, that is, at $x = 2$. We check for differentiability at $x = 2$, using k instead of the usual h, in order to avoid confusion with the function $h(x)$.

$\lim_{k \to 0} \frac{h(2+k) - h(2)}{k}$

$= \lim_{k \to 0} \frac{\left[\sqrt[3]{3(2+k) - 6} + 5\right] - 5}{k}$

$= \lim_{k \to 0} \frac{\sqrt[3]{3k}}{k}$

$= \sqrt[3]{3} \lim_{k \to 0} \frac{1}{k^{\frac{2}{3}}}$

$= \infty$

The function has a vertical tangent at $x = 2$. It is differentiable for all reals except $x = 2$

$$\lim_{h\to 0^-}\frac{y(0+h)-y(0)}{h}$$
$$=\lim_{h\to 0^-}\frac{\sqrt[3]{|h|}-0}{h}$$
$$=\lim_{h\to 0^-}\frac{-\sqrt[3]{|h|}}{h}$$
$$==\lim_{h\to 0^-}\left(-\frac{1}{h^{\frac{2}{3}}}\right)$$
$$=-\infty$$
$$\lim_{h\to 0^+}\frac{y(0+h)-y(0)}{h}$$
$$=\lim_{h\to 0^+}\frac{\sqrt[3]{|h|}-0}{h}$$
$$=\lim_{h\to 0^+}\frac{\sqrt[3]{|h|}}{h}$$
$$==\lim_{h\to 0^+}\left(\frac{1}{h^{\frac{2}{3}}}\right)$$
$$=\infty$$

The problem is a cusp.

19.
Note that the sine function is off, so

$$P(x)$$
$$=\sin\left(|x|\right)-1$$
$$=\begin{cases}-\sin x-1, & x\langle 0\\ \sin x-1, & x\geq 0\end{cases}$$

The graph of $P(x)$ has a corner at $x = 0$. The function is differentiable for all reals except $x = 0$.

20.
Since the cosine function is even, so

$$Q(x)$$
$$=3\cos\left(|x|\right)$$
$$=3\cos x$$

The function is differentiable for all reals.

21.
The function is piecewise-defined in terms of polynomials, so it is differentiable everywhere except possibly at $x = 0$ and at $x = 3$. Check $x = 0$:

$$\lim_{h\to 0^-}\frac{g(0+h)-g(0)}{h}$$
$$=\lim_{h\to 0^-}\frac{(h+1)^2-(1)}{h}$$
$$=\lim_{h\to 0^-}\frac{h^2+2h}{h}$$
$$=\lim_{h\to 0^-}(h+2)$$
$$=2$$
$$\lim_{h\to 0^+}\frac{g(0+h)-g(0)}{h}$$
$$=\lim_{h\to 0^+}\frac{(2h+1)-1}{h}$$
$$=\lim_{h\to 0^+}2$$
$$=2$$

The function is differentiable at $x = 0$.
Check $x = 3$:
Since $g(3) = (4 - 3)^2 = 1$ and

$$\lim_{h\to 3^-}g(x)$$
$$=\lim_{h\to 3^-}(2x+1)$$
$$=2(3)+1$$
$$=7$$

The function is not continuous (and hence not differentiable) at $x = 3$.
The function is differentiable for all reals except $x = 3$.

22.
Note that

$$C(x)$$
$$=x|x|$$
$$=\begin{cases}-x^2, & x\langle 0\\ x^2, & x\ge 0\end{cases}$$

so it is differentiable for all x except possibly at $x = 0$.
Check $x = 0$:

$$\lim_{h\to 0}\frac{C(0+h)-C(0)}{h}$$
$$=\lim_{h\to 0}\frac{h|h|-0}{h}$$
$$=\lim_{h\to 0}|h|$$
$$=0$$

The function is differentiable for all reals.