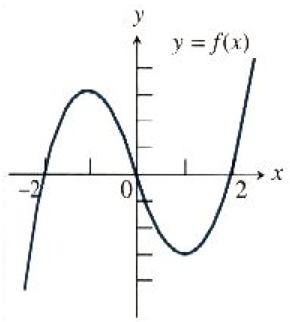
MCV4U1 **Worksheet 4.3**

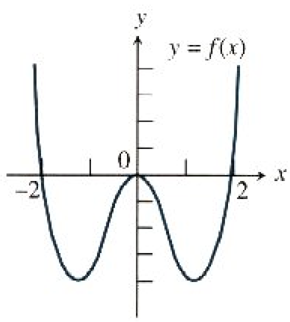
This worksheet serves as an additional exercise to complement the lesson and the examples given. Worksheets may take more than one day to complete. If you are stuck, read again the notes taken in class, see the additional examples from the website, work with fellow students, come to the tutor lessons after school. Complete solutions to all questions are available on the website. When in doubt → do more!

1. Use the graph of the function *f* to estimate where f' and f'' are 0, positive, and negative.

   a.

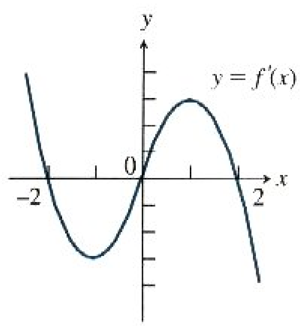


   b.

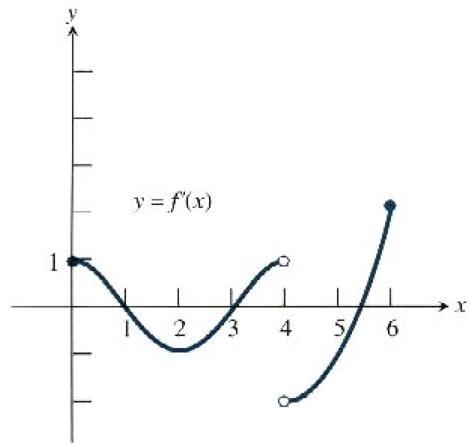


2. Use the graph of f' to estimate the intervals on which the function *f* is **(a)** increasing or **(b)** decreasing. **(c)** Estimate where *f* has local extreme values.

   a.



   b.



3. Use analytic methods to find the intervals on which the function is **(a)** increasing, **(b)** decreasing, **(c)** concave up, **(d)** concave down. Then find any **(e)** local extreme values, **(f)** inflection points. Support your answers graphically.

   a. $y = -2x^3 + 6x^2 - 3$

   b. $y = xe^{\frac{1}{x}}$

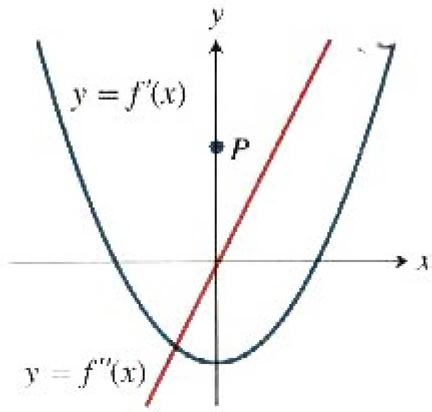
   c. $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

   d. $y = \dfrac{x}{x^2 + 1}$

4. Use the derivative of the function $y' = (x-1)^2(x-2)$ to find the points at which *f* has a

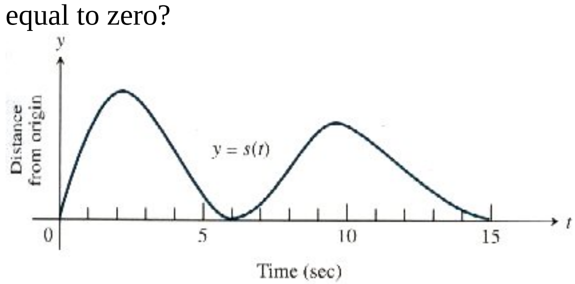
   a. local maximum,
   b. local minimum,
   c. point of inflection.

5. Shown are the graphs of the first and second derivatives of a function $y = f(x)$. Copy the figure and add a sketch of a possible graph of *f* that passes through the point *P*.



6. A particle is moving along a line with position function $s(t)$. Find the (a) velocity and (b) acceleration, and (c) describe the motion of the particle for $t \geq 0$.
   a. $s(t) = t^2 - 4t + 3$
   b. $s(t) = t^3 - 3t + 3$

7. The graph of the position function $y = s(t)$ of a particle moving along a line is given. At approximately what times is the particle's (a) velocity equal to zero? (b) acceleration equal to zero?



8. If $f(x)$ is a differential function and $f'(c) = 0$ at an interior point *c* of *f*'s domain, must *f* have a local maximum or minimum at $x = c$? Explain.

9. If $f(x)$ is a twice-differentiable function and $f''(c) = 0$ at an interior point *c* of *f*'s domain, must *f* have an inflection point at $x = c$? Explain.

10. *Connecting f, f', and f"*. Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

$f(-2) = 8$	$f'(x) > 0$ for $\lvert x \rvert > 2$
$f(0) = 4$	$f'(x) < 0$ for $\lvert x \rvert < 2$
$f(2) = 0$	$f''(x) < 0$ for $x < 0$
$f'(2) = f'(-2) = 0$	$f''(x) > 0$ for $x > 0$

11. *Using Behaviour to Sketch*. Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Curve
$x < 2$		falling, concave up
2	1	horizontal tangent
$2 < x < 4$		rising, concave up
4	4	inflection point
$4 < x < 6$		rising, concave down
6	7	horizontal tangent
$x > 6$		falling, concave down

MCV4U1 **Worksheet 4.3 Solutions**

1.

a. *f':*

Zero: $x = \pm 1$;
positive: $(-\infty, -1)$ and $(1, \infty)$;
negative: $(-1, 1)$

f":

Zero: $x = 0$;
positive: $(0, \infty)$;
negative: $(-\infty, 0)$

b.

f':

Zero: $x \approx 0, \pm 1.25$;
positive: $(-1.25, 0)$ and $(1.25, \infty)$;
negative: $(-\infty, -1.25)$ and $(0, 1.25)$

f":

Zero: $x \approx \pm 0.7$;
positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
negative: $(-0.7, 0.7)$

2.

a.

(a) $(-\infty, -2]$ and $[0, 2]$

(b) $[-2, 0]$ and $[2, \infty)$

(c) Local maxima: $x = -2$ and $x = 2$;
local minimum: $x = 0$

b.

(a) $[0, 1]$, $[3, 4]$, and $[5.5, 6]$

(b) $[1, 3]$ and $[4, 5.5]$

(c) Local maxima: $x = 1$, $x = 4$
(if f is continuous at $x = 4$), and $x = 6$;
local minima: $x = 0$, $x = 3$, and $x = 5.5$

3.

a.

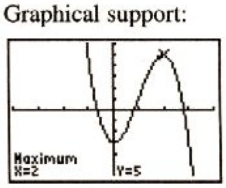
$y' = -6x^2 + 12x = -6x(x - 2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

$y'' = -12x + 12 = -12(x - 1)$

Intervals	$x < 1$	$x > 1$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

Graphical support:



$[-4, 4]$ by $[-6, 6]$

(a) $[0, 2]$

(b) $(-\infty, 0]$ and $[2, \infty)$

(c) $(-\infty, 1)$

(d) $(1, \infty)$

(e) Local maximum: $(2, 5)$;
local minimum: $(0, -3)$

(f) At $(1, 1)$

b.

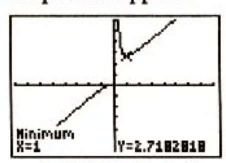
$y' = xe^{1/x}(-x^{-2}) + e^{1/x} = e^{1/x}\left(1 - \frac{1}{x}\right)$

Intervals	$x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	$+$	$-$	$+$
Behavior of y	Increasing	Decreasing	Increasing

$y'' = e^{1/x}(x^{-2}) + \left(1 - \frac{1}{x}\right)e^{1/x}(-x^{-2}) = \frac{e^{1/x}}{x^3}$

Intervals	$x < 0$	$x > 0$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

Graphical support:



$[-8, 8]$ by $[-6, 6]$

(a) $(-\infty, 0)$ and $[1, \infty)$

(b) $(0, 1]$

(c) $(0, \infty)$

(d) $(-\infty, 0)$

(e) Local minimum: $(1, e)$

(f) None

c.

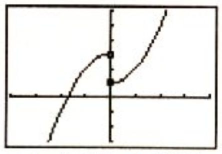
$$y' = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$$

Intervals	$x < 0$	$x > 0$
Sign of y'	$+$	$+$
Behavior of y	Increasing	Increasing

$$y'' = \begin{cases} -2, & x < 0 \\ 2, & x > 0 \end{cases}$$

Intervals	$x < 0$	$x > 0$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

Graphical support:



$[-4, 4]$ by $[-3, 6]$

(a) $(-\infty, 0)$ and $[0, \infty)$

(b) None

(c) $(0, \infty)$

(d) $(-\infty, 0)$

(e) Local minimum: $(0, 1)$

(f) Note that $(0, 1)$ is not an inflection point because the graph has no tangent line at this point.

There are no inflection points.

d.

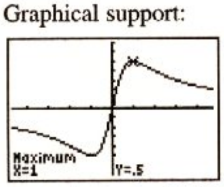
$$y' = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

Intervals	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

$$\begin{aligned} y'' &= \frac{(x^2+1)^2(-2x) - (-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4} \\ &= \frac{(x^2+1)(-2x) - 4x(-x^2+1)}{(x^2+1)^3} \\ &= \frac{2x^3 - 6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3} \end{aligned}$$

Intervals	$x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x$
Sign of y''	$-$	$+$	$-$	$+$
Behavior of y	Concave down	Concave up	Concave down	Concave up

Graphical support:



$[-4.7, 4.7]$ by $[-0.7, 0.7]$

(a) $[-1, 1]$

(b) $(-\infty, -1]$ and $[1, \infty)$

(c) $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$

(d) $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$

(e) Local maximum: $\left(1, \frac{1}{2}\right)$;

local minimum: $\left(-1, -\frac{1}{2}\right)$

(f) $(0, 0)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$, and $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$

4.

$y' = (x-1)^2(x-2)$

Intervals	$x < 1$	$1 < x < 2$	$2 < x$
Sign of y'	$-$	$-$	$+$
Behavior of y	Decreasing	Decreasing	Increasing

$$\begin{aligned} y'' &= (x-1)^2(1) + (x-2)(2)(x-1) \\ &= (x-1)[(x-1) + 2(x-2)] \\ &= (x-1)(3x-5) \end{aligned}$$

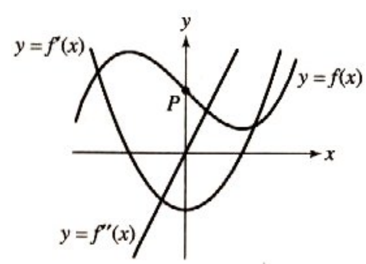
Intervals	$x < 1$	$1 < x < \frac{5}{3}$	$\frac{5}{3} < x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

(a) There are no local maxima.

(b) There is a local (and absolute) minimum at $x = 2$.

(c) There are points of inflection at $x = 1$ and at $x = \frac{5}{3}$.

5.



6.

a.

(a) $v(t) = s'(t) = 2t - 4$

(b) $a(t) = v'(t) = 2$

(c) It begins at position 3 moving in a negative direction. It moves to position -1 when $t = 2$, and then changes direction, moving in a positive direction thereafter.

b.

(a) $v(t) = s'(t) = 3t^2 - 3$

(b) $a(t) = v'(t) = 6t$

(c) It begins at position 3 moving in a negative direction. It moves to position 1 when $t = 1$, and then changes direction, moving in a positive direction thereafter.

7.

(a) The velocity is zero when the tangent line is horizontal, at approximately $t = 2.2$, $t = 6$, and $t = 9.8$.

(b) The acceleration is zero at the inflection points, approximately $t = 4$, $t = 8$, and $t = 11$.

8.

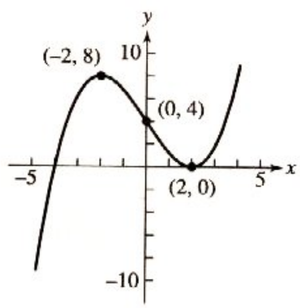
No. f must have a horizontal tangent at that point, but f could be increasing (or decreasing), and there would be no local extremum. For example, if $f(x) = x^3$, $f'(0) = 0$ but there is no local extremum at $x = 0$.

9.

No. $f''(x)$ could still be positive (or negative) on both sides of $x = c$, in which case the concavity of the function would not change at $x = c$. For example, if $f(x) = x^4$, then $f''(0) = 0$, but f has no inflection point at $x = 0$.

10.

One possible answer:



11.

One possible answer: