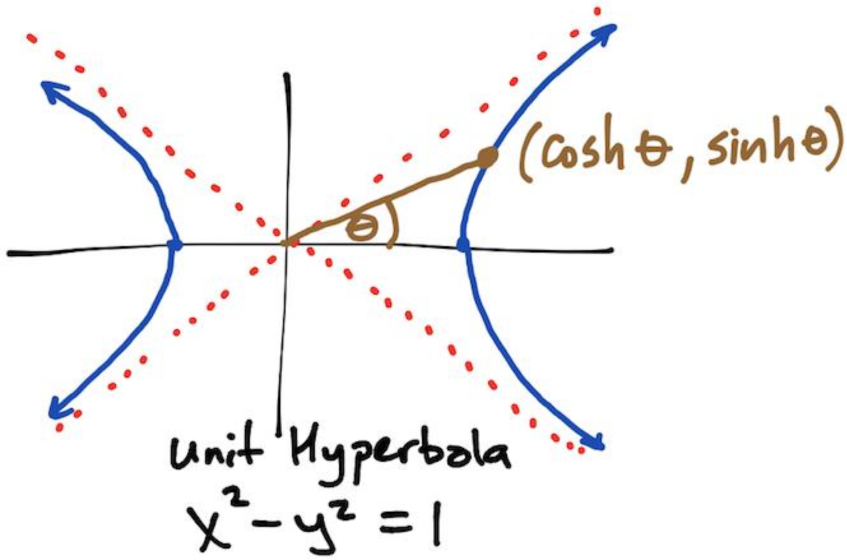
§10.6—Hyperbolic Functions

The circle has its trig functions, and the hyperbola has, what are known as, hyperbolic functions. On the Unit Circle, any point along the circle has the coordinate $\left(\cos q, \sin q\right)$. On a branch of the Unit Hyperbola, any point has the coordinate $\left(\cosh q, \sinh q\right)$.



Guess what the "h" is for . . .

We read $\cosh q$ as "hyperbolic cosine of theta," and $\sinh q$ is similarly read "hyperbolic sine of theta." Just as the circular trig functions show up in many real-world applications, so do the hyperbolic trig functions. In fact, many applications of exponential functions are really hyperbolic trig functions in disguise.

Because we will be talking about the hyperbolic **functions**, we will use *x* as the input, rather than q.

Definition of the Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\operatorname{csch} x = \dfrac{1}{\sinh x}$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\operatorname{sech} x = \dfrac{1}{\cosh x}$
$\tanh x = \dfrac{\sinh x}{\cosh x}$	$\coth x = \dfrac{1}{\tanh x}$

Notice that the functions $f(x)=\sinh x$ and $f(x)=\cosh x$ are the differences and the sums, respectively, of the two exponential functions $y=\frac{1}{2}e^{x}$ and $y=\frac{1}{2}e^{-x}$. Because of this, the graphs of $f(x)=\sinh x$ and $f(x)=\cosh x$ can be obtained by subtracting and adding the ordinates of the two exponential graphs.

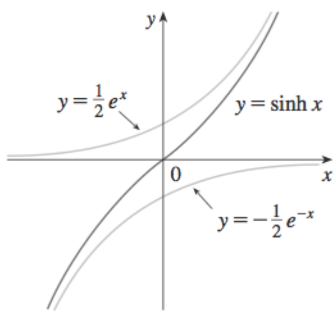


FIGURE 1
$y=\sinh x=\frac{1}{2}e^{x}-\frac{1}{2}e^{-x}$

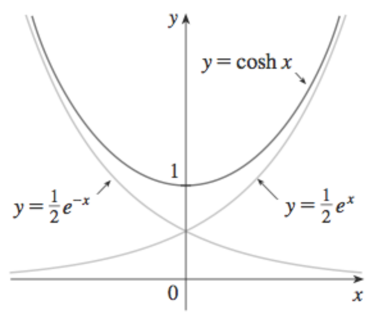


FIGURE 2
$y=\cosh x=\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}$

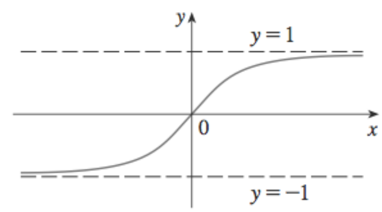
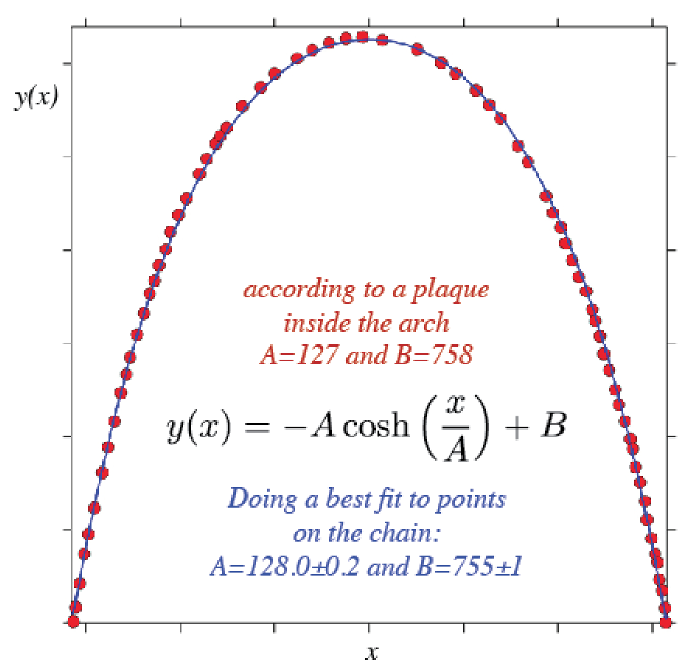


FIGURE 3
$y=\tanh x$

Example 1:
Find the domain and range and any symmetry for the three hyperbolic functions shown above.

Notice how the graph of $y=\cosh x$ resembles a parabola. This mistaken identity is quite easy to make, especially without quantitative analysis. The graph of $y=\cosh x$ is actually called a **catenary curve**, from the Latin *catena*, meaning "chain." This is because a heavy chain (or cable) suspended between two fixed points at the same elevation will take the sagging shape of a catenary with equation $y=a\cosh\left(\frac{x}{a}\right)$.

The most famous catenary (and mistaken parabola) of them all is the St. Louis/Gateway Arch.



Example 2:
Using the definition of $y = \cosh x$ and $y = \sinh x$, simplify $\cosh^2 x - \sinh^2 x$.

Just as there are many circular trig identities (and proofs), so there are many hyperbolic trig identities. For a list of many more, click here.

Let's talk calculus:

Example 3:
Using the definitions, find the derivatives of $y = \sinh x$ and $y = \cosh x$.

Example 4:
Using the definition, find the derivative of $y = \tanh x$.

Here are the derivatives of the Hyperbolic Functions

$$\frac{d}{dx}\left[\sinh u\right]=\left(\cosh u\right)u' \qquad \frac{d}{dx}\left[\coth u\right]=-\left(\operatorname{csch}^2 u\right)u'$$

$$\frac{d}{dx}\left[\cosh u\right]=\left(\sinh u\right)u' \qquad \frac{d}{dx}\left[\operatorname{sech} u\right]=-\left(\operatorname{sech} u\tanh u\right)u'$$

$$\frac{d}{dx}\left[\tanh u\right]=\left(\operatorname{sech}^2 u\right)u' \qquad \frac{d}{dx}\left[\operatorname{csch} u\right]=-\left(\operatorname{csch} u\coth u\right)u'$$

Example 5:

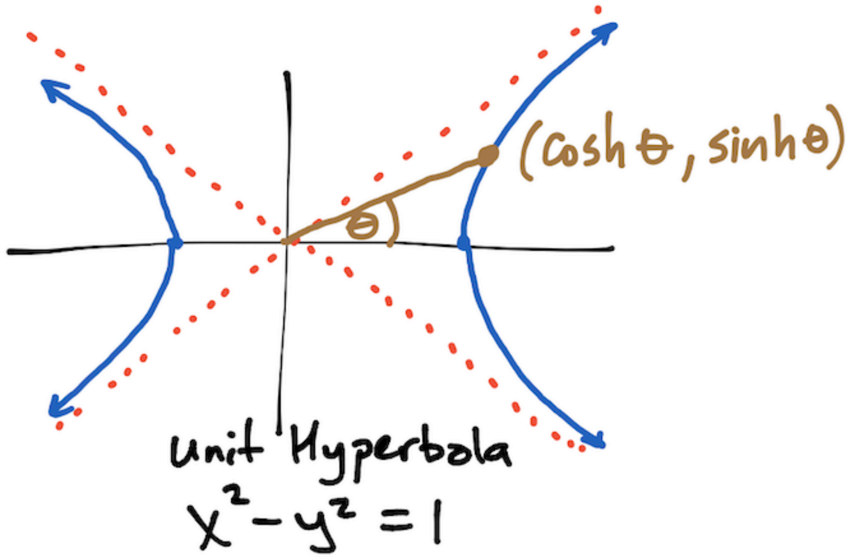
(a) $\frac{d}{dx}\left[\sinh\left(x^2-3\right)\right]=$ (b) $\frac{d}{dx}\left[\ln\left(\cosh x\right)\right]=$ (c) $\frac{d}{dx}\left[x\sinh x-\cosh x\right]=$

Example 6:

Evaluate $\int \cosh 2x \sinh^2 2x\, dx =$

§12.5—Hyperbolic Functions

The circle has its trig functions, and the hyperbola has, what are known as, hyperbolic functions. On the Unit Circle, any point along the circle has the coordinate $(\cos\theta, \sin\theta)$. On a branch of the Unit Hyperbola, any point has the coordinate $(\cosh\theta, \sinh\theta)$.



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$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

Notice that the functions $f(x) = \sinh x$ and $f(x) = \cosh x$ are the differences and the sums, respectively, of the two exponential functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$. Because of this, the graphs of $f(x) = \sinh x$ and $f(x) = \cosh x$ can be obtained by subtracting and adding the ordinates of the two exponential graphs.

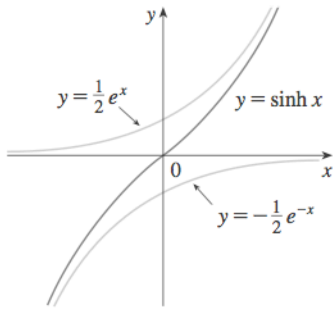

FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

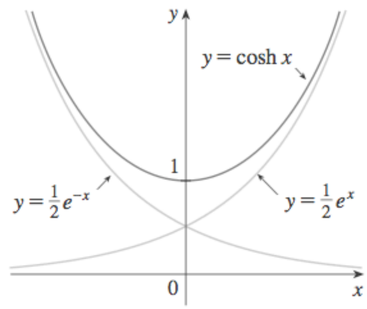

FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

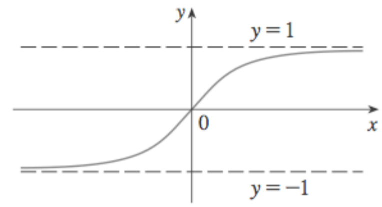

FIGURE 3
$y = \tanh x$

Example 1:
Find the domain and range and any symmetry for the three hyperbolic functions shown above.

$f(x) = \sinh x$

$D_f : \mathbb{R}$

$R_f : \mathbb{R}$

odd function

so $\sinh(-x) = -\sinh x$

$g(x) = \cosh x$

$D_g : \mathbb{R}$

$R_g : [1, \infty)$

Even Function

so $\cosh(-x) = \cosh x$

$h(x) = \tanh x$

$D_h : \mathbb{R}$

$R_h : (-1, 1)$

odd function

so $\tanh(-x) = -\tanh x$

Notice how the graph of $y = \cosh x$ resembles a parabola. This mistaken identity is quite easy to make, especially without quantitative analysis. The graph of $y = \cosh x$ is actually called a **catenary curve**, from the Latin *catena*, meaning "chain." This is because a heavy chain (or cable) suspended between two fixed points at the same elevation will take the sagging shape of a catenary with equation $y = a\cosh\left(\frac{x}{a}\right)$.

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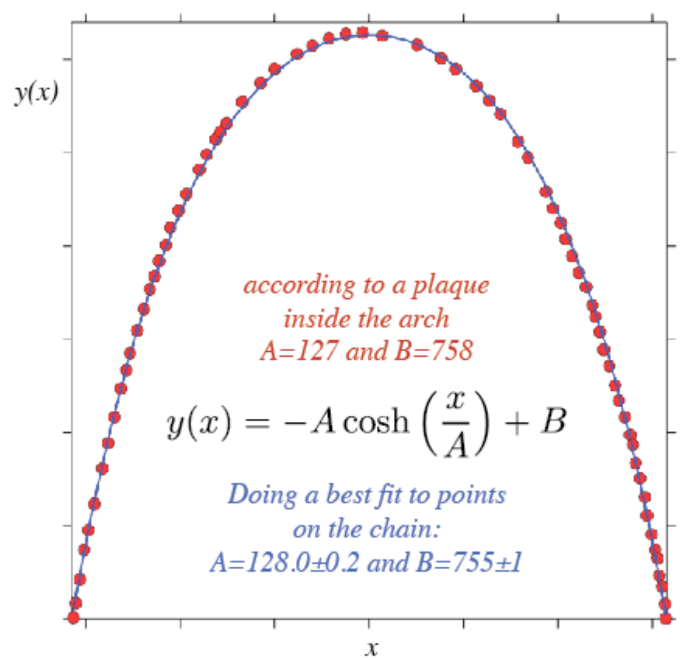

Example 2:

Using the definition of $y=\cosh x$ and $y=\sinh x$, simplify $\cosh^2 x-\sinh^2 x$.

$$(\cosh x)^2-(\sinh x)^2$$

$$\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$\frac{e^{2x}+2+e^{-2x}}{4}-\frac{e^{2x}-2+e^{-2x}}{4}$$

$$\frac{\cancel{e^{2x}}+2+\cancel{e^{-2x}}-\cancel{e^{2x}}+2-\cancel{e^{-2x}}}{4}$$

$$\frac{4}{4}$$

$$\boxed{1}$$

So $\boxed{\cosh^2 x-\sinh^2 x=1}$

Papa PID for Hyperbolic Functions

Just as there are many circular trig identities (and proofs), so there are many hyperbolic trig identities. For a list of many more, click here.

Let's talk calculus:

Example 3:

Using the definitions, find the derivatives of $y=\sinh x$ and $y=\cosh x$.

$$y=\sinh x$$
$$y=\tfrac{1}{2}(e^x-e^{-x})$$
$$y'=\tfrac{1}{2}(e^x+e^{-x})$$
$$y'=\cosh x$$

So $\boxed{\frac{d}{dx}\sinh x=\cosh x}$

$$y=\cosh x$$
$$y=\tfrac{1}{2}(e^x+e^{-x})$$
$$y'=\tfrac{1}{2}(e^x-e^{-x})$$
$$y'=\sinh x$$

So $\boxed{\frac{d}{dx}[\cosh x]=\sinh x}$

Example 4:

Using the definition, find the derivative of $y=\tanh x$.

$$y=\tanh x$$
$$y=\frac{\sinh x}{\cosh x}$$
$$y'=\frac{(\cosh x)(\cosh x)-(\sinh x)(\sinh x)}{\cosh^2 x}$$
$$y'=\frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}$$
$$y'=\frac{1}{\cosh^2 x}$$
$$y'=\operatorname{sech}^2 x$$

So $\boxed{\frac{d}{dx}[\tanh x]=\operatorname{sech}^2 x}$

Here are the derivatives of the Hyperbolic Functions

$$\frac{d}{dx}\left[\sinh u\right]=\left(\cosh u\right)u' \qquad \frac{d}{dx}\left[\coth u\right]=-\left(\operatorname{csch}^2 u\right)u'$$

$$\frac{d}{dx}\left[\cosh u\right]=\left(\sinh u\right)u' \qquad \frac{d}{dx}\left[\operatorname{sech} u\right]=-\left(\operatorname{sech} u\tanh u\right)u'$$

$$\frac{d}{dx}\left[\tanh u\right]=\left(\operatorname{sech}^2 u\right)u' \qquad \frac{d}{dx}\left[\operatorname{csch} u\right]=-\left(\operatorname{csch} u\coth u\right)u'$$

Example 5:

(a) $\frac{d}{dx}\left[\sinh\left(x^2-3\right)\right]=$

$\cosh(x^2-3)(2x)$

$2x\cosh(x^2-3)$

(b) $\frac{d}{dx}\left[\ln\left(\cosh x\right)\right]=$

$\frac{1}{\cosh x}\cdot(\sinh x)$

$\frac{\sinh x}{\cosh x}$

$\tanh x$

(c) $\frac{d}{dx}\left[x\sinh x-\cosh x\right]=$

$1\cdot\sinh x+x\cdot\cosh x-\sinh x$

$x\cosh x$

Example 6:

Evaluate $\int \cosh 2x\sinh^2 2x\,dx=$

$\int(\cosh(2x))\cdot(\sinh(2x))^2dx$

off by 2

$\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\sinh(2x)\right)^3+C$

correction

$\boxed{\frac{1}{6}\sinh^3(2x)+C}$