Worksheet 7.3—Separable Differential Equations
Show all work. No Calculator unless specified.

Multiple Choice

1. (OK, so you can use your calculator right away on a non-calculator worksheet. Use it on this one.) A sample of Kk-1234 (an isotope of Kulmakorpium) loses 99% of its radioactive matter in 199 hours. What is the half-life of Kk-1234?
   (A) 4 hours      (B) 6 hours      (C) 30 hours      (D) 100.5 hours     (E) 143 hours

2. In which of the following models is \( \frac{dy}{dt} \) directly proportional to \( y \)?
   I. \( y = e^{kt} + C \)
   II. \( y = Ce^{kt} \)
   III. \( y = 28^{kt} \)
   IV. \( y = 3 \left( \frac{1}{2} \right)^{3t+1} \)
   (A) I only    (B) II only    (C) I and II only    (D) II and III only    (E) II, III, and IV    (F) all of them
3. (Use your calculator on this one, too, but get the exact answer first.) The rate at which acreage is being consumed by a plot of kudzu is proportional to the number of acres already consumed at time \( t \). If there are 2 acres consumed when \( t = 1 \) and 3 acres consumed when \( t = 5 \), how many acres will be consumed when \( t = 8 \)?

(A) 3.750   (B) 4.000   (C) 4.066   (D) 4.132   (E) 4.600

Free Response

For problems 4 – 13, find the general solution to the following differential equations, then find the particular solution using the initial condition.

4. \( \frac{dy}{dx} = \frac{x}{y}, \; y(1) = -2 \)  
5. \( \frac{dy}{dx} = -\frac{x}{y}, \; y(4) = 3 \)  
6. \( \frac{dy}{dx} = \frac{y}{x}, \; y(2) = 2 \)

7. \( \frac{dy}{dx} = 2xy, \; y(0) = -3 \)  
8. \( \frac{dy}{dx} = (y + 5)(x + 2), \; y(0) = -1 \)  
9. \( \frac{dy}{dx} = \cos^2 y, \; y(0) = 0 \)
10. \[ \frac{dy}{dx} = (\cos x)e^{y+\sin x}, \ y(0) = 0 \]

11. \[ \frac{dy}{dx} = e^{x-y}, \ y(0) = 2 \]

12. \[ \frac{dy}{dx} = -2xy^2, \ y(1) = 0.25 \]

13. \[ \frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}, \ y(e) = 1 \]

For problems 14 – 17, find the solution of the differential equation \( \frac{dy}{dt} = ky \) that satisfies the given conditions.

14. \( k = 1.5, \ y(0) = 100 \)

15. \( k = -0.5, \ y(0) = 200 \)

16. \( y(0) = 50, \ y(5) = 100 \)

17. \( y(1) = 55, \ y(10) = 30 \) (divide one by the other)
18. AP 2010B-5 (No Calculator)

Consider the differential equation \( \frac{dy}{dx} = \frac{x + 1}{y} \).

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for \(-1 < x < 1\), sketch the solution curve that passes through the point \((0, -1)\).

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the \(xy\)-plane for which \(y \neq 0\). Describe all points in the \(xy\)-plane, \(y \neq 0\), for which \(\frac{dy}{dx} = -1\).

(c) Find the particular solution \(y = f(x)\) to the given differential equation with the initial condition \(f(0) = -2\).
19. AP 2006-5

Consider the differential equation \( \frac{dy}{dx} = \frac{1+y}{x} \), where \( x \neq 0 \).

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.

(b) Find the particular solution \( y = f(x) \) to the differential equation with the initial condition \( f(-1) = 1 \) and state its domain.
20. AP 2005-6

Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.

(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of $f$ at $(1, -1)$ and use it to approximate $f(1.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$. 
0. \[ y = 100(0.01) e^{(0.01)x}, \] where 100 = initial amount.  
\[ 50 = 100(0.01)^{2/199} \]
\[ \frac{1}{2} = (0.01)^{2/199} \]
\[ \ln (1/2) = \frac{2}{199} \ln (0.01) \]
\[ t = \frac{199 \ln (1/2)}{\ln (0.01)} = 29.952 \]
\[ t \approx 29.952 \times 30 = 300 \text{ hr} \]

1. \( \frac{dy}{dt} = ky \) gives solutions of form \[ y = C e^{kt} \] or \[ y = e^{kt} + C \]
   - I. \( y = e^{kt} + C \rightarrow \text{NO} \)
   - II. \( y = C e^{kt} \rightarrow \text{YES} \)
   - III. \( y = 28 e^{k/2} = (e^{28} e^{k/2}) \rightarrow \text{YES} \)
   - IV. \( y = 3(\frac{1}{2})^{x/2} = 3(e^{x/2})^{3/2} \rightarrow \text{YES} \)

2. \[ \frac{dn}{dt} = 4.8 h \] gives solutions of form \[ n = C e^{kt} \] or \[ n = e^{kt} + C \]
   - I. \( n = e^{kt} + C \rightarrow \text{NO} \)
   - II. \( n = C e^{kt} \rightarrow \text{YES} \)
   - III. \( n = 28 e^{kt} = (e^{28} e^{kt}) \rightarrow \text{YES} \)
   - IV. \( n = 3(\frac{1}{2})^{x/2} = 3(e^{x/2})^{3/2} \rightarrow \text{YES} \)

3. \[ y = C e^{kt}, \ (1/2), \ (5/3) \]
   - I. \( y = C e^{kt} \)
   - II. \( y = \frac{1}{2} (C e^{kt}) \)
   - III. \( y = \frac{5}{3} (C e^{kt}) \)
   - IV. \( y = 2 (C e^{kt}) \)

4. \[ \frac{dy}{dx} = \frac{x}{y}, \ y(1) = -2 \]
   - \[ ay + by^2 + c \]
   - \[ y = \pm \sqrt{x^2 + c} \]
   - \[ a + (b, c) \]
   - \[ y = -\sqrt{x^2 + c} \]

5. \[ \frac{dy}{dx} = -\frac{y}{x}, \ y(4) = 3 \]
   - \[ ay^2 + by + c \]
   - \[ y = \pm \sqrt{25 - x^2} \]

6. \[ \frac{dy}{dx} = \frac{y}{x}, \ y(2) = 2 \]
   - \[ ay + by^2 + c \]
   - \[ y = e^{\ln|x|} + c \]

7. \[ \frac{dy}{dx} = 2xy, \ y(1) = -3 \]
   - \[ ay^2 + by + c \]
   - \[ y = e^{2x^2}, \ y = -e^{2x^2} \]

8. \[ \frac{dy}{dx} = 4(x+5)(x+2), \ y(0) = -1 \]
   - \[ ay^2 + by + c \]
   - \[ y = e^{x^2 + 2x}, \ y = -e^{x^2 + 2x} \]

9. \[ y = e^{4(x^2 + 2x)}, \ y(0) = 5 \]
   - \[ ay^2 + by + c \]
   - \[ y = e^{4x^2 + 8x}, \ y = -e^{4x^2 + 8x} \]

10. \[ \frac{dy}{dx} = \ln(y+1), \ y(0) = -1 \]
    - \[ ay + by^2 + c \]
    - \[ y = e^{\ln|x|} + c \]
    - \[ y = e^{\ln|x|} + c \]
9. \( \frac{dy}{dx} = \cos^2 y \), \( y(0) = 0 \)
   \[ \int \sec^2 y \, dy = \int dx \]
   \[ \tan y = x + c \]
   \[ y = \arctan(x + c) \]

at \((0,0); \) \( D = \arctan C \)
\[ C = \tan 0 = 0 \]
so \( y = \arctan x \).

10. \( \frac{dy}{dx} = (\sin x)e^{y+\sin x} \), \( y(0) = 0 \)
   \[ \frac{dy}{dx} = \cos x \cdot e^y \cdot \sin x \]
   \[ \int e^{-y} \, dy = \int \cos x \cdot e^x \, dx \]
   \[-e^{-y} = e^x + C \]
   \[ e^{-y} = e^x - C \]
   \[ y = \ln(C - e^x) \]

   \[ y = -\ln(2 - e^x) \]

11. \( \frac{dy}{dx} = e^{x-y} \), \( y(0) = 2 \)
   \[ \frac{dy}{dx} = e^x \cdot e^{-y} \]
   \[ e^y \, dy = e^x \, dx \]
   \[ e^y = e^x + C \]
   \[ y = \ln(e^x + C) \]

at \((0,2); \) \( 2 = \ln(1+C) \)
\[ 1 + C = e^2 \]
so \( y = \ln(e^x + e^2 - 1) \).

12. \( \frac{dy}{dx} = -2xy^2 \), \( y(1) = 0.25 \)
   \[ \int y^{-2} \, dy = \int -2x \, dx \]
   \[ \frac{1}{y} = -x^2 + C \]
   \[ y = \frac{1}{-x^2 + C} \]

at \((1,1/4); \) \( \frac{1}{4} = 1 + C \)
so \( y = \frac{1}{x^2 + 3} \).

13. \( \frac{dy}{dt} = ky \rightarrow y = Ce^{kt} \)
   \[ k = 1.5, y(0) = 100 \]
\[ y = 100e^{1.5t} \]

14. \( \frac{dy}{dt} = ky \rightarrow y = Ce^{kt} \)
   \[ k = -0.5, y(0) = 200 \]
\[ y = 200e^{-0.5t} \]

15. \( y = Ce^{kt} \), \( y(0) = 55, y(10) = 30 \)
   \[ 55 = Ce^{k(10)} \]
   \[ 30 = Ce^{k} \]
   \[ C = 55 \left( \frac{30}{55} \right)^{-1/10} \]
   \[ y = 55 \left( \frac{30}{55} \right)^{-1/10} e^{(30/55)k} \]
   \[ y = 55 \left( \frac{30}{55} \right)^{-1/10} e^{k/5} \]
   \[ y = 55 \left( \frac{30}{55} \right)^{-1/10} \text{ or } y = 55 \left( \frac{30}{55} \right)^{-1/10} e^{(30/55)k} \]

16. \( \frac{dy}{dt} = ky \rightarrow y = Ce^{kt} \)
   \[ y(0) = 50, y(5) = 100 \]
   \[ \frac{1}{2} = e^{5k}, \ k = \frac{\ln(1/2)}{5} \]
   \[ y = 50e^{(1/2)k} \]
   \[ \text{or } y = 50 \left( \frac{1}{2} \right) \]
\( \frac{dv}{dt} = -2v - 32, v(0) = -50 \)

(a) \( \frac{dv}{dt} = -2(v + 16) \)
\[ \frac{1}{v+16} \frac{dv}{dt} = -2 \frac{dt}{dt} \]
\[ \ln|v+16| = -2t + C \]
\[ v+16 = e^{-2t}e^C \]
\[ v+16 = C e^{-2t} \]
\[ v = Ce^{-2t} - 16 \]

So \( v(t) = -34e^{-2t} - 16 \)

(b) \[ \lim_{t \to \infty} \left( -\frac{34}{e^{2t}} - 16 \right) = 0 - 16 = -16 \text{ ft/sec} \]

(c) \[ v(t) = -20 \]
\[ -20 = -34e^{-2t} - 16 \]
\[ 4 = 34e^{-2t} \]
\[ e^{-2t} = \frac{4}{34} \]
\[ -2t = \ln\left(\frac{2}{17}\right) \]
\[ t = -\frac{1}{2} \ln\left(\frac{2}{17}\right) \text{ sec} \]

The slowest speed she reaches with her parachute is 16 ft per second.

(d) \[ \int v(t)dt = \int \left( -\frac{34}{e^{2t}} - 16 \right) dt \]
\[ h(t) = 17e^{-2t} - 16t + C \]
if \( h(0) = 2000 \)
\[ 2000 = 17 + C, C = 1983 \]
\[ h(t) = 17e^{-2t} - 16t + 1983 \]
\[ h(t) = 0 \text{ from calculator when } t = 123.9375 \text{ sec} \]

So her speed at the time she hits the ground is 16 ft/sec, so it is safe for her to touch down.

\[ \frac{dy}{dx} = \frac{x+1}{y} \]

(b) \[ \frac{dy}{dx} = -1 \]
when \( x + 1 = y \)
\[ y = -x - 1 \]
\[ \frac{dy}{dx} = -1 \text{ for all points satisfying } y = -x - 1 \text{ such that } y \neq 0. \]

(c) \[ \frac{dy}{dx} = \frac{x+1}{y}, \quad f(0) = -2 \]
\[ \int y \, dy = \int (x+1) \, dx \]
\[ \frac{1}{2}y^2 = \frac{1}{2}x^2 + x + C \]
\[ y^2 = x^2 + 2x + C \]
\[ y = \pm \sqrt{x^2 + 2x + C} \]
\[ \text{Given: } a + (0r2) = -2 = \pm \sqrt{C} \]
\[ C = 4 \]
\[ y = -\sqrt{x^2 + 2x + 4} \]
20. \( \frac{dy}{dx} = \frac{1+y}{x}, x \neq 0 \)

(a)

(b) \( \frac{dy}{dx} = \frac{1+y}{x}, f(-1) = 1 \)
- \( y + 1 = C|x| \), since C can be ±
- \( y = C|x| - 1 \)
- \( + (-1, 1); \) x
- 1 = C - 1 - 1
- 1 = C - 1, C = 2
- \( y = 2|x| - 1 \)
- The graph of the solution is
- So Domain is \( D_y = \mathbb{R} \times x \mid x < 0 \)

21. \( \frac{dy}{dx} = -\frac{2x}{y} \)

(a)

(b) \( f(1) = -1 \)
- \( \frac{dy}{dx} \mid (1, 1) = 2 \)
- \( y + 1 = 2(x - 1) \)
- \( f(1, 1) \approx y(1, 1) \)
- \( = -1 + 2(1, -1) \)
- \( = -1 + 2(-1) \)
- \( = -1 - 2 \)
- \( = -3 \) \( \checkmark \)

(c) \( \frac{dy}{dx} = -\frac{2x}{y}, f(1) = -1 \)
- \( y dy = \int -2x \ dx \)
- \( \int \frac{1}{2} y^2 = -x^2 + C \)
- \( y^2 = -2x^2 + C \)
- \( y = \pm \sqrt{-2x^2 + C} \)
- \( + (1, 1); \) x
- \( 1 = \pm \sqrt{-2 + C} \)
- \( 1 = -2 + C, C = 3 \)
- So \( y = -\sqrt{-2x^2 + 3} \)
- or \( y = -\sqrt{3 - 2x^2} \) \( \checkmark \)